

A new in-place truncated Fourier transform algorithm

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Abstract

Let \mathcal{R} be a ring containing a principal $N = (2^k)^{\text{th}}$ root of unity. We present an algorithm which, given a polynomial $f(z)$ over \mathcal{R} of degree less than $n \leq N$, gives a vector of n weighted evaluations of $f(z)$, using $\frac{1}{2}n \log n + \mathcal{O}(n)$ ring multiplications. This algorithm requires only the space for the input itself and additional $\mathcal{O}(1)$ ring elements and bounded-precision integers. The algorithm uses a linear-time method of breaking $f(z)$ into reduced images modulo polynomials of the form $z^m + 1$, m a power of two, then uses the Fast Fourier Transform (FFT) to get a vector of evaluation points of each image. The result is an in-place truncated Fourier transform with complexity comparable to the truncated Fourier transform of Van der Hoeven. Using this algorithm we give an in-place algorithm for polynomial multiplication requiring space for only the inputs and output and an additional $\mathcal{O}(1)$ bounded-precision integers and ring elements.

1 Introduction

The Fast Fourier Transform (FFT) is an algorithm which maps a polynomial $f(z)$ over a ring \mathcal{R} to a vector comprised of n evaluations of $f(z)$ using $\mathcal{O}(n \log n)$ arithmetic ring operations. This algorithm lends itself to fast integer and polynomial multiplication. The radix-2 Cooley-Tukey FFT [1] requires that the input have a power-of-two size. This results in jumps in space and time complexity at power-of-two sized inputs. Truncated Fourier Transform (TFT) algorithms by Van der Hoeven [6] and by Harvey and Roche [4] have addressed these jumps in time and space complexity, respectively.

In this paper we present a new in-place TFT algorithm that improves on the complexity of the Harvey-Roche TFT, in terms of the number of ring multiplications, by a constant factor. By in-place, we mean that the algorithm, acting on a size- n input, writes its result in place of its input and only requires space for the output itself plus an additional $\mathcal{O}(1)$ ring elements and integers bounded by cn for a fixed constant c . We then show how this algorithm may be used towards polynomial multiplication.

In this section we describe the FFT and the previous TFT algorithms.

1.1 The discrete Fourier transform and its inverse

The *discrete Fourier transform* (DFT) of a polynomial $f(z)$ is its vector of evaluations at the distinct powers of a root of unity. Specifically, if $f(z) = \sum_{i=0}^{n-1} a_i z^i$ and \mathcal{R} contains an n^{th} primitive root of unity ω , then we define the discrete Fourier transform of $f(z)$ as

$$\text{DFT}_{\omega}^n(f) = (f(\omega^j))_{0 \leq j < n}. \quad (1)$$

We treat the vector $a = (a_0, a_1, \dots, a_{n-1})$ and f as equivalent and use $\text{DFT}_{\omega}^n(a)$ and $\text{DFT}_{\omega}^n(f)$ interchangeably. The map $\text{DFT}_{\omega}^n : \mathcal{R}[z]/(z^n - 1) \rightarrow \mathcal{R}^n$ forms an additive group isomorphism. If ω is a *principal* root of unity, that is, for j not divisible by n , $\sum_{i=0}^{n-1} \omega^{ij} = 0$, then we can obtain $f \bmod (z^n - 1)$ by way of its inverse map, $\text{IDFT}_{\omega}^n : \mathcal{R}^n \rightarrow \mathcal{R}[z]/(z^n - 1)$, defined by

$$\text{IDFT}_{\omega}^n(\hat{a}_0, \dots, \hat{a}_n) = \frac{1}{n} \text{DFT}_{\omega^{-1}}^n(\hat{a}_0, \dots, \hat{a}_n), \quad (2)$$

where we again we treat a polynomial as equivalent to its vector of coefficients. The DFT gives a polynomial multiplication algorithm:

Theorem 1 (The Convolution Theorem).

$$fg \bmod (z^n - 1) = \text{IDFT}_{\omega}^n(\text{DFT}_{\omega}(f) \cdot \text{DFT}_{\omega}(g)), \quad (3)$$

where “ \cdot ” is the vector dot-product, and, given two polynomial $g(z), h(z) \in \mathcal{R}[z]$, $g(z) \bmod h(z)$ denotes the unique polynomial $r(z)$ such that $h(z)$ divides $g(z) - r(z)$ and $\deg(r) < \deg(h)$ throughout.

1.2 The Fast Fourier Transform

We can compute the discrete Fourier transform of $f(z)$, f reduced modulo $(z^n - 1)$, by way of the *Fast Fourier Transform* (FFT), which recursively reduces a DFT into smaller DFTs based on the factorization of n . The *radix-2* FFT of Cooley and Tukey [1], computes a DFT for n a power-of-two by a divide-and-conquer approach. We break $f(z)$ into two polynomials $f_0(z)$ and $f_1(z)$, where

$$f_0(z) = \sum_{k=0}^{n/2-1} a_{2k} z^k, \quad f_1(z) = \sum_{k=0}^{n/2-1} a_{2k+1} z^k. \quad (4)$$

Note that $f(z) = f_0(z^2) + z f_1(z^2)$. Thus, as $\omega^{j+n/2} = -\omega^j$, and $\omega^{2j} = \omega^{2(j+n/2)}$ for an n^{th} root of unity ω , we have that

$$f(\omega^j) = f_0(\omega^{2j}) + \omega^j f_1(\omega^{2j}), \quad f(\omega^{j+n/2}) = f_0(\omega^{2j}) - \omega^j f_1(\omega^{2j}), \quad (5)$$

for $0 \leq j < n/2$. The pair of computations (5) are known as a *butterfly operation*. We perform these butterfly operations in place. Note that the values $f_i(\omega^{2j})$ comprise $\text{DFT}_{\omega^2}(f_i)$ for $i = 0, 1$, two DFTs of size $n/2$. Thus, to compute the discrete Fourier transform of f , we split f into polynomials f_0 and

f_1 , compute their size- $n/2$ DFTs with root-of-unity ω^2 , and then compute the butterfly operations (5), for $0 \leq j < n$, to obtain the evaluations of f at the powers of ω .

If the butterfly operations are performed in place, the resulting evaluations $f(\omega^j)$ will be written in *bit-reversed* order. More precisely, if we let $\text{rev}_c(j)$ denote the integer resulting from reversing the first c bits of j , we have that $f(\omega^j)$ will be written in place of a_k , where $k = \text{rev}_{\log(n)}(j)$ and \log is taken to be base-2 throughout. As an example,

$$\text{rev}_5(11) = \text{rev}_5(01011_2) = 11010_2 = 16 + 8 + 2 = 26. \quad (6)$$

Procedure FFT(\mathbf{a} , n , ω), an in-place implementation of the radix-2 Cooley-Tukey FFT

Input:

- $\mathbf{a} = (\mathbf{a}(0), \dots, \mathbf{a}(n-1))$, an array containing a vector $a \in \mathcal{R}^n$.
- ω , a root of $z^n - 1$, for n a power of two.

Result: $\text{DFT}_\omega^n(a)$ is written to the array \mathbf{a} in bit-reversed order.

```

1 for  $i \leftarrow 1$  to  $\log_2(n)$  do
2    $m \leftarrow n/2^i$ 
   // Complete  $m$  DFTs of size  $n/m$ 
3    $(\gamma, \zeta) \leftarrow (1, \omega^m)$ 
4   for  $j \leftarrow 0$  to  $\frac{n}{2m} - 1$  do
5      $l \leftarrow \text{rev}_i(j)m$ 
6     for  $k \leftarrow 0$  to  $m-1$  do
7        $(l_1, l_2) \leftarrow (l+k, l+k+m)$ 
       // Butterfly operations (5)
8        $(\mathbf{a}(l_1), \mathbf{a}(l_2)) \leftarrow (\mathbf{a}(l_1) + \gamma\mathbf{a}(l_2), \mathbf{a}(l_1) - \gamma\mathbf{a}(l_2))$ 
9      $\gamma \leftarrow \gamma\zeta$ 

```

Procedures FFT and IFFT describe implementations of the in-place FFT and its inverse, made non-recursive so as to make them in-place. Normally one would pre-compute powers of ω and store them in an array in bit-reversed order. To avoid this space overhead we perform the butterfly operations in an order such that we can compute the required powers of ω^m sequentially. For better memory performance, one can produce the powers of ω in an order that allows us to traverse the array in sequential order, at the cost of an additional $\mathcal{O}(n)$ ring multiplications.

The elements γ , ω , ζ , and array \mathbf{a} comprise all the ring elements in the algorithm. Our only ring operations are due to the butterfly operations (line 8) and computing the powers of ω (lines 3, 9). ω^m can be computed by a square-and-multiply approach using $\mathcal{O}(\log m)$ multiplications. Enumerating these operations gives us the following cost:

Procedure IFFT(\mathbf{a}, n, ω), an in place implementation of the inverse-FFT

Input: $\mathbf{a} = (\mathbf{a}(0), \dots, \mathbf{a}(n-1))$, an array containing $\text{DFT}_\omega^n(a)$, for a vector $a \in \mathcal{R}^n$, for n a power of two, in bit-reversed order.

Result: The vector a is written to array \mathbf{a} in place of its DFT.

```

1 for  $i \leftarrow 1$  to  $\log_2(n)$  do
2    $m \leftarrow 2^{i-1}$ 
   // Complete  $\frac{n}{2m}$  DFTs of length  $2m$  with root of unity  $\omega^{-1}$ 
3    $(\gamma, \zeta) \leftarrow (1, \omega^{-m})$ 
4   for  $j \leftarrow 0$  to  $\frac{n}{2m} - 1$  do
5     for  $k \leftarrow 0$  to  $m$  do
6        $l \leftarrow 2km$ 
7        $(l_1, l_2) \leftarrow (j + l, j + l + m)$ 
       // Butterfly operations (5)
8        $(\mathbf{a}(l_1), \mathbf{a}(l_2)) \leftarrow (\mathbf{a}(l_1) + \gamma \mathbf{a}(l_2), \mathbf{a}(l_1) - \gamma \mathbf{a}(l_2))$ 
9      $\gamma \leftarrow \gamma \zeta$ 
10 for  $i \leftarrow 0$  to  $n - 1$  do  $\mathbf{a}(i) \leftarrow \frac{1}{n} \mathbf{a}(i)$ 

```

Lemma 2. Let n be a power of two and let \mathbf{a} be an array containing the coefficients of a polynomial $f(z)$ reduced modulo $(z^n - 1)$. Computing $\text{DFT}_\omega^n(a)$ by way of procedure FFT, requires $n \log n + \mathcal{O}(1)$ additions and $\frac{1}{2}n \log n + \mathcal{O}(n)$ multiplications in \mathcal{R} . Procedure IFFT has similar complexity.

The approach described for n a power of two generalizes for n a prime power or a product of small primes, though in practise n is almost always chosen to be a power-of-2.

Using the radix-2 FFT, if $d = \deg(fg)$, we choose n to be the least power of 2 exceeding d . This entails appending zeros to the input arrays a and b . By this method, computing a product of degree 2^k requires the same time and space cost as computing a product of degree $2^k + m$, for $0 \leq m < 2^k$. It would be preferable to have a radix-2 FFT algorithm whose time cost grows smoothly relative to n , and which requires $n + \mathcal{O}(1)$ space for arbitrary integers $n > 0$, and not just for n a power of two. Crandall's *devil's convolution* algorithm [2] somewhat flattens these jumps in complexity, though not entirely. It works by reducing a discrete convolution of arbitrary length into more easily computable convolutions. More recently, truncated Fourier transform (TFT) algorithms, described hereafter, have addressed this issue.

1.3 Truncated Fourier Transform algorithms

The *Truncated Fourier Transform* (TFT) algorithm due to Van der Hoeven [6] returns a pruned DFT such that the result contains as few evaluations as is necessary. Suppose $\deg(fg) < n$ and $N = 2^k$ where $k = \lceil \log_2 n \rceil$. If ω is an N^{th}

primitive root of unity, we define the TFT of $f(z)$ with parameters ω and n to be the vector of evaluation points

$$\text{TFT}_\omega^n(f) = \left(f(\omega^{\text{rev}_k(0)}), f(\omega^{\text{rev}_k(1)}), \dots, f(\omega^{\text{rev}_k(n-1)}) \right). \quad (7)$$

It comprises the first n terms of $\text{DFT}_\omega^n(f)$, written in bit-reversed order. The TFT algorithm works by discarding unnecessary butterfly operations (5) where at least one of its inputs are 0. The inverse transform is more involved. Given a length- N vector a , the inverse TFT takes $\text{TFT}_\omega^n((a_0, \dots, a_{n-1}))$ and the non-transformed vector coefficients $(a_{n+1}, \dots, a_{N-1})$ and returns the entire vector a . For our purposes a is the vector of coefficients of f and a_l is presumed to be zero for $l \geq n$. The algorithm relies on the fact that one can obtain any two values in a butterfly relation (5) given the other two values by solving a 2×2 linear system. This method still requires space for $N + \mathcal{O}(1)$ ring elements.

Theorem 3 (Van der Hoeven). *The TFT algorithm computes $\text{TFT}_\omega^n(f)$ using $n\lceil \log n \rceil + \mathcal{O}(n)$ ring additions and $\frac{1}{2}(n\lceil \log n \rceil) + \mathcal{O}(n)$ multiplications by powers of ω . The inverse TFT algorithm requires similarly many multiplications by powers of ω , and $n\lceil \log n \rceil + \mathcal{O}(n)$ **shifted** ring additions.*

A *shifted addition* is merely the combined operation of an addition plus, possibly, a multiplication by $2^{\pm 1}$. For floating-point numbers and $x \in \mathbb{Z}/p\mathbb{Z}$, multiplication by $2^{\pm 1}$, using bit shifts and bit operations, has bit-complexity comparable to addition. We note that Van der Hoeven computed more precise bounds on the number of ring operations (i.e. without “big-Oh” notation). We chose to write the cost as is to simplify the expressions in terms of n . Van der Hoeven’s algorithm also generalizes to allow us to compute arbitrary subsets of DFTs. We define, for a set $I \subset \{0, 1, \dots, N-1\}$,

$$\text{TFT}_\omega^I(f) = \left(f(\omega^{\text{rev}_k(i)}) \right)_{i \in I}. \quad (8)$$

By this notation $\text{TFT}_\omega^n = \text{TFT}_\omega^I$ for $I = \{0, 1, \dots, n-1\}$.

More recently, Harvey and Roche [4] developed an in-place variant of the TFT which writes TFT_ω^n in place of $(a_i)_{0 \leq i < n}$. This algorithm was used towards fast in-place polynomial multiplication. This algorithm is implemented iteratively to avoid using an additional $\mathcal{O}(\log n)$ space for the stack, but the algorithm effectively has a recursive structure. We first recursively compute

$$\text{TFT}_{\omega^2}^{\lceil n/2 \rceil}(f_0) = f_0(\omega^{2\text{rev}_k(i)}), \quad 0 \leq i \leq \lceil n/2 \rceil, \quad (9)$$

in place of the subset of coefficients of f , $(a_0, a_2, \dots, a_{\lceil n/2 \rceil})$. Then, if n is odd, we compute $\gamma = f_1(\omega^{2l})$, where $l = \text{rev}_k(\lceil n/2 \rceil)$. Such an evaluation takes $\mathcal{O}(n)$ ring operations. We then compute

$$\text{TFT}_{\omega^2}^{\lfloor n/2 \rfloor}(f_1) = f_1(\omega^{2\text{rev}_k(i)}), \quad 0 \leq i \leq \lfloor n/2 \rfloor, \quad (10)$$

in place of the remaining coefficients of f_1 , $(a_1, a_3, \dots, a_{\lfloor n/2 \rfloor})$. These two smaller TFTs, and γ if n is odd, gives us the information required to complete

the butterfly relations needed to compute $\text{TFT}_\omega^n(f)$. These additional linear-time evaluations make the Harvey-Roche truncated Fourier transform require, at worst, a constant factor additional ring operations.

Theorem 4 (Roche [5], Chapter 3). *The in-place truncated Fourier transform requires at most $\frac{5}{6}n\lceil\log n\rceil + \frac{n-1}{3}$ ring multiplications to compute $\text{TFT}_\omega^n(f)$.*

1.4 The Half-DFT

In the TFT algorithm of this paper we will use the *half-DFT* (HDFT). For ω , a root of $z^n + 1$, the half-DFT is the map defined by

$$\text{HDFT}_\omega^n(f) = (f(\omega^1), f(\omega^3), \dots, f(\omega^{2^{n-1}})). \quad (11)$$

It is the evaluation of f at the roots of $z^n + 1$. The half-DFT can be computed by way of a DFT [3]. Namely, if we let a' be the vector defined by $a'_j = a_j\omega^j$ for $0 \leq j < n$, then, up to re-ordering,

$$\text{HDFT}_\omega^n(a) = \text{DFT}_{\omega^2}^n(a'). \quad (12)$$

Procedure **HalfDFT** describes an implementation of this approach.

Procedure HalfDFT(\mathbf{a} , n , ω), an in-place algorithm for computing the half-DFT

Input:

- $\mathbf{a} = (\mathbf{a}(0), \dots, \mathbf{a}(n-1))$, an array containing a vector $a \in \mathcal{R}^n$.
- ω , a root of $z^n + 1$, where n is a power of two.

Result: The half-DFT $\text{HDFT}_\omega^n(a)$ is written in place of a

```

1  $\gamma \leftarrow 1$ 
2 for  $i \leftarrow 0$  to  $n-1$  do
3    $\mathbf{a}(i) \leftarrow \gamma \mathbf{a}(i)$ 
4    $\gamma \leftarrow \gamma \omega$ 
5 FFT( $\mathbf{a}$ ,  $n$ ,  $\omega^2$ )
```

Procedure InverseHalfDFT(\mathbf{a} , n , ω), an in-place algorithm for inverting the half-DFT

Input:

- $\mathbf{a} = (\mathbf{a}(0), \dots, \mathbf{a}(n-1))$, an array containing $\text{HalfDFT}_\omega^n(a)$ for some $a \in \mathcal{R}^n$.
- ω , a root of $z^n + 1$, where n is a power of two.

Result: a is written in place of $\text{HDFT}_\omega^n(a)$

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1 IFFT( $\mathbf{a}$ ,  $n$ ,  $\omega^2$ )
2  $\gamma \leftarrow 1$ 
3 for  $i \leftarrow 0$  to  $n-1$  do
4    $\mathbf{a}(i) \leftarrow \gamma \mathbf{a}(i)$ 
5    $\gamma \leftarrow \gamma \omega^{-1}$ 
```

Inverting the half-DFT is straightforward: we apply an inverse FFT and then multiply the i^{th} coefficient by ω^{-i} . We have the following complexities.

Lemma 5. *Let n be a power of two. $\text{HalfDFT}(\mathbf{a}, n, \omega)$ and $\text{InverseHalfDFT}(\mathbf{a}, n, \omega)$ both require $n \log n + \mathcal{O}(1)$ additions, and $\frac{1}{2}n \log_2 n + \mathcal{O}(n)$ multiplications.*

The cost of $\text{HalfDFT}(\mathbf{a}, n, \gamma)$ is merely the cost of an FFT plus an additional $2n$ ring multiplications.

2 An in-place TFT algorithm

2.1 Description of the result of the algorithm and notation

We use the following notation throughout section 2 and thereafter. Suppose now that we have a polynomial $f(z) = \sum_{j=0}^{n-1} a_j z^j \in \mathcal{R}[z]$ of degree at most $n-1$, where n can be any positive integer. Write n as $n = \sum_{i=1}^s n_i$, where $n_i = 2^{n(i)}$, $n(i) \in \mathbb{Z}_{\geq 0}$ and $n_i > n_j$ for $1 \leq i < j \leq s$. For $1 \leq i \leq s$, we let

$$\Phi_i = z^{n_i} + 1 \quad \text{and} \quad \Gamma_i = \prod_{j=1}^i \Phi_j. \quad (13)$$

Note that $\Phi_i \bmod \Phi_j = 2$ and $\Gamma_i \bmod \Phi_j = 2^i$ for $j \geq i$. Our transform algorithm will compute the evaluations of $f(z)$ at the roots of Φ_i . Namely, if we fix a canonical root $\omega = \omega_1$ of Φ_1 , and then let, for $2 \leq i \leq s$, $\omega_i = \omega_1^{n(1)/n(s)}$, a root of Φ_i , we will compute the vector of evaluations

$$f(\omega_i^j) \quad \text{for} \quad 0 \leq j < n_i, \quad 1 \leq i \leq s. \quad (14)$$

If we let

$$I(n) = \{0 \leq k < n : \text{for some } i, 1 \leq i \leq s, n_i \leq k < 2n_i\}, \quad (15)$$

then the set of evaluations we compute is exactly $\text{TFT}_{\omega}^{I(n)}(f)$.

The roots of unity used as evaluation points in $\text{TFT}_{\omega}^{I(n)}(f)$ will differ from those used in $\text{TFT}_{\omega}^n(f)$. For instance, TFT_{ω}^n will always use all roots-of-unity of order dividing n , whereas $\text{TFT}_{\omega}^{I(n)}$ will instead use all roots of unity of order $2n_1$.

We define, for $1 \leq i \leq s$, the images

$$f_i = f \bmod \Phi_i \quad \text{and} \quad f_i^* = 2^{-i+1} f_i. \quad (16)$$

We call the polynomials f_i^* the *weighted images* of f . We will first compute f_i^* as they are less costly to compute than the images f_i . Once we have all the weighted images f_i^* we will reweigh them to obtain the unweighted images f_i . We also define, for $1 \leq i \leq s$, the images

$$C_i = f \bmod \Gamma_i. \quad (17)$$

We call the polynomials C_i the *combined images* of f . We also use

$$q_i = \begin{cases} f & \text{if } i = 0 \\ f \bmod \Gamma_i & \text{if } 1 \leq i \leq s \end{cases} \quad (18)$$

the quotient produced by dividing f by Γ_i and

$$n_i^* = n \bmod n_i. \quad (19)$$

It is straightforward to obtain q_i , given f . Note that the degrees of any two distinct terms of Γ_i differ by at least n_i , and that $\deg(q_i) < n_i$. Thus, as Γ_i is monic, we have that the coefficients of q_i merely comprise the coefficients of the higher-degree terms of f . More precisely,

$$q_i = a_{n-1}z^{(n_i^*-1)} + a_{n-2}z^{(n_i^*-2)} + \cdots + a_{n-n_i^*+1}z + a_{n-n_i^*}. \quad (20)$$

By a similar argument, we also have that, for $1 < i \leq s$,

$$q_i = q_{i-1} \text{ quo } \Phi_i. \quad (21)$$

We note that $q_s = 0$. We will also use the remainder resulting from dividing q_i by Φ_i ,

$$r_i = q_{i-1} \bmod \Phi_i, \quad 1 \leq i \leq s. \quad (22)$$

Our transform has three steps: we first iteratively break f in-place into the remainders r_i ; we then iteratively write f_i^* in place of r_i for $i = 1, 2, \dots, s$; lastly we reweigh the weighted images f_i^* to get f_i and perform a half-DFT on each image f_i separately to give us the weighted evaluation points.

2.2 Breaking f into the sequence of remainders r_1, \dots, r_s

We first break $f = q_0$ into its quotient and remainder dividing by $z^{n_1} + 1$,

$$r_1 = f \bmod (z^{n_1} + 1) = \sum_{i=0}^{n_1-1} (a_i - a_{i+n_1}) z^i, \quad (23)$$

$$q_1 = f \text{ quo } (z^{n_1} + 1) = \sum_{i=0}^{n_1^*-1} a_{i+n_1} z^i, \quad (24)$$

where $a_i = 0$ for $i \geq n$. This can be done in place with n_1^* subtractions in \mathcal{R} . We then similarly break q_1 into r_2 and q_2 , then q_2 into r_3 and q_3 , and continue until we have r_1, \dots, r_{s-1} and q_{s-1} . Since $\deg(q_{s-1}) < n_s = \deg(\Phi_s)$, r_s is exactly q_{s-1} .

This process is completely reversible. From $r_i = q_{i-1} \bmod \Phi_i$ and $q_i = q_{i-1} \text{ quo } \Phi_i$ we can easily obtain q_{i-1} . If we write $r_i = \sum_{j=1}^{n_i-1} b_j z^j$ and $q_i = \sum_{j=0}^{n_i^*-1} c_j z^j$, then

$$\begin{aligned} q_{i-1} = & (b_0 + c_0) + (b_1 + c_1)z + \cdots + (b_{n_i^*-1} + c_{n_i^*-1})z^{n_i^*-1} + \\ & b_{n_i^*} z^{n_i^*} + \cdots + b_{n_i-1} z^{n_i-1} + \\ & c_0 z^{n_i} + \cdots + c_{n_i^*} z^{n_i^*-1}. \end{aligned} \quad (25)$$

As $r_s = q_{s-1} \bmod \Phi_s = q_{s-1}$, we can obtain q_{s-1} from r_{s-1} and r_s . We then can obtain q_j from q_{j+1} and r_{j+1} until we have $q_0 = f$.

2.3 Writing the weighted images f_i^* in place of the remainders r_i

We first note that f_1^* is precisely r_1 . We will iteratively produce the remaining weighted images.

Suppose, at the start of the i^{th} iteration, we have f_1^*, \dots, f_i^* , and r_{i+1}, \dots, r_s , and we want to write f_{i+1}^* in place of r_{i+1} . We have

$$f_{i+1}^* = 2^{-i} f \bmod \Phi_{i+1}, \quad (26)$$

$$= 2^{-i} (\Gamma_i q_i + C_i) \bmod \Phi_{i+1}, \quad (27)$$

$$= (q_i + 2^{-i} C_i) \bmod \Phi_{i+1}, \quad (28)$$

$$= (r_{i+1} + (2^{-i} C_i \bmod \Phi_{i+1})) \quad (29)$$

Unfortunately, we don't have the combined image C_i , but rather the weighted images f_j^* , $1 \leq j \leq i$, from which we can reconstruct C_i . We would like to be able to compute $C_i \bmod \Phi_{i+1}$ in place from the weighted images f_j^* . To that end, the following lemma gives us a basis for $C_i \bmod \Phi_{i+1}$.

Given an integer e , we will let $e[i]$ refer to the i^{th} bit of e , i.e.

$$e = \sum_{i=0}^{\lfloor \log(e) \rfloor} e[i] 2^i, \quad e[i] \in \{0, 1\}. \quad (30)$$

Lemma 6. Fix i, j and k , $1 \leq j \leq i < k \leq s$. Suppose that $f_j^* = z^e$ and $f_l^* = 0$ for $l \neq j$, $0 \leq l \leq i$. Then $C_i \bmod \Phi_k$ is nonzero only if $e[n(l)] = 1$ for $j < l \leq i$, in which case

$$C_i \bmod \Phi_k = 2^{i-1} z^e \bmod \Phi_k. \quad (31)$$

Given that $z^{n_k} \bmod \Phi_k = -1$, we have that

$$2^{i-1} z^e \bmod \Phi_k = (-1)^{e[n(k)]} 2^{i-1} z^{(e \bmod n_k)}, \quad (32)$$

where $e \bmod n_k$ is the integer e^* such that $n_k | (e - e^*)$ and $0 \leq e^* < n_k$. The values $e[n(l)]$ can be determined from $n_l = 2^{n(l)}$ and e by way of a bitwise AND operation.

Example 7. Suppose $n = 86 = 64 + 16 + 4 + 2$, and suppose that

$$f_1^* = f \bmod z^{64} + 1 = z^e, \quad f_2^* = 0, \quad f_3^* = 0,$$

and $\deg(f) < 64 + 16 + 4$. Then one can check that $C_3 \bmod (z^2 + 1)$, where $C_3 = f \bmod [(z^{64} + 1)(z^{16} + 1)(z^4 + 1)]$, satisfies

$$C_3 \bmod (z^2 + 1) = \begin{cases} 0 & \text{if } e \in [0, 20) \cup [24, 28) \cup [32, 52) \cup [56, 60), \\ 4 & \text{if } e = 20, 28, 52, \text{ or } 56, \\ 4z & \text{if } e = 21, 29, 53, \text{ or } 57, \\ -4 & \text{if } e = 22, 30, 54, \text{ or } 58, \\ -4z & \text{if } e = 23, 31, 55, \text{ or } 59. \end{cases}$$

Our lemma is stated more generally than what is needed for the algorithm, as we only need a result for $C_i \bmod \Phi_{i+1}$ for $1 \leq i < s$. However, in doing so this allows us to prove the lemma inductively.

Proof of lemma 6. We fix e and j , and prove by induction on i .

Base case: First consider the base case $i = j$, in which case the non-zero criterion of the lemma always holds and we need only show (31). We have that

$$C_i \equiv 2^{i-1} f_i^* \equiv 2^{i-1} z^e \pmod{\Phi_i}, \quad (33)$$

and, by Chinese remaindering,

$$C_i = C_{i-1} + \Gamma_{i-1} \left(\Gamma_{i-1}^{-1} (2^{i-1} z^e - C_{i-1}) \bmod \Phi_i \right). \quad (34)$$

Since $f \bmod \Phi_l = 0$ for $1 \leq l < i$, it follows that $C_{i-1} = 0$, and

$$C_i = \Gamma_{i-1} z^e \bmod \Gamma_i. \quad (35)$$

As $e < n_i$, $\Gamma_{i-1} z^e$ is already necessarily reduced modulo Γ_i , and we have C_i is exactly $\Gamma_{i-1} z^e$. Reducing this modulo Φ_k , we have

$$C_i \bmod \Phi_k = 2^{i-1} z^e \bmod \Phi_k \quad (36)$$

as desired.

Inductive step: Suppose now that the lemma holds for a fixed $i \geq j$, and consider $C_{i+1} \bmod \Phi_k$, $k > i + 1$. We suppose that $f_l^* = 0$ for $1 \leq l \neq j \leq i + 1$ and $f_j^* = z^e$. As $f \bmod \Phi_{i+1} = 0$, Chinese remaindering gives us

$$C_{i+1} = C_i - \Gamma_i \left(\Gamma_i^{-1} C_i \bmod \Phi_{i+1} \right). \quad (37)$$

We prove the inductive step by cases.

Case 1: If $e[n(l)] = 0$ for some $l, j < l \leq i$, then by the induction hypothesis, $C_i \bmod \Phi_{i+1} = 0$. Thus, given (37), $C_{i+1} = C_i$ and $C_{i+1} \bmod \Phi_k = C_i \bmod \Phi_k$. By the induction hypothesis again, $C_i \bmod \Phi_k = 0$, completing the proof for this case.

Case 2: If $e[n(l)] = 1$ for $j < l \leq i$, then by the induction hypothesis and (32),

$$C_i \bmod \Phi_{i+1} = 2^{i-1} z^e \bmod \Phi_{i+1} = 2^{i-1} (-1)^{e[n(i+1)]} z^{(e \bmod n_{i+1})}, \quad (38)$$

applying this to (37) gives

$$C_{i+1} = C_i - \Gamma_i \left((-1)^{e[n(i+1)]} z^{(e \bmod n_{i+1})} \right) \bmod \Gamma_{i+1}. \quad (39)$$

By inspection of the degrees of the polynomials appearing in (39), we have equality:

$$C_{i+1} = C_i - \Gamma_i \left((-1)^{e[n(i+1)]} z^{(e \bmod n_{i+1})} \right). \quad (40)$$

Reducing this modulo Φ_k and applying the induction hypothesis to $C_i \bmod \Phi_k$ gives us

$$C_{i+1} \bmod \Phi_k = 2^{i-1} z^e - 2^{i-1} (-1)^{e[n(i+1)]} z^{(e \bmod n_{i+1})} \bmod \Phi_k \quad (41)$$

$$= 2^{i-1} \left(1 - (-1)^{e[n(i+1)]} \right) z^{(e \bmod n_{i+1})} \bmod \Phi_k, \quad (42)$$

as $z^e \equiv z^{e \bmod n_{i+1}} \pmod{\Phi_k}$. Since $1 - (-1)^{e[n(i+1)]}$ evaluates to 2 if $e[n(i+1)] = 1$ and 0 otherwise, this completes the proof. \square

Lemma 6 tells us the *contribution* of f_i^* towards subsequent images. If e satisfies the non-zero criterion of the lemma, then by (29), a term $c_{j,e} z^e$ of f_j^* , $j \leq i$ will contribute $\frac{1}{2} (-1)^{e[n(i+1)]} z^{(e \bmod n_{i+1})}$ to f_{i+1}^* . In order to make the contributions have weight ± 1 , we instead first reweigh r_i by 2 and compute $2f_i^*$, and then divide by 2 thereafter. **AddContributions** adds the contribution of f_1^*, \dots, f_{i-1}^* to f_i^* . It also can subtract these contributions, for the purposes of the inverse TFT algorithm.

According to lemma 6, f_j^* , $0 \leq j \leq i$, only a proportion of 2^{i-j} of the terms of f_j^* will have a non-zero contribution to f_{i+1}^* . Thus the total cost of adding contributions of f_j^* towards f_i^* , $i > j$, is less than $2\#f_j^* = 2n_j$. It follows that the total additions and subtractions in \mathcal{R} required to add all these contributions is bounded by $2n$. Since **AddContributions** only scales array ring elements by ± 1 , we have the following complexity:

Lemma 8. *Let $n = \sum_{i=1}^s n_i$. Calling **AddContributions**($\mathbf{a}, n, i, \text{add?}$) for $1 \leq i \leq s$ entails no more than $2n$ ring additions and no ring multiplications.*

In the manner we have chosen to add these contributions, we will have to make $s-1$ passes through our array to add them all. One way we could avoid this is to instead add all the contributions from f_1^* , and then all the contributions from f_2^* , and so forth, adding up all the contributions from a single term at once. We could use that a term $c_{j,e} z^e$ of f_j^* that does not contribute towards f_{i+1}^* will not contribute to f_k^* for any $k > i$. This would reduce the number of passes we make through the larger portion of the array, though the cache performance of potentially writing to $s-1$ images at once raises questions.

When adding contributions to f_i^* , any two terms $c_{j,e} z^e$ and $c_{j,e^*} z^{e^*}$ of f_j^* whose bits $e[n(l)], e^*[n(l)]$ agree for $j < l < i$ will both contribute to f_i^* in the same fashion (i.e. we will either add or subtract both coefficients $c_{i,e}$ and c_{i,e^*} to an image, or do nothing). Thus we need only inspect the non-zero criterion of one exponent e in a block of exponents $kn_{i-1} \leq e < (k+1)n_{i-1}$. Similarly, we need only inspect one exponent in a contiguous block of n_i exponents in order to determine their shared value of $(-1)^{e[n(i)]}$.

Given an exponent that does not satisfy the non-zero criterion, our implementation will generate the next exponent that does satisfy the non-zero criterion, by way of bit operations.

Procedure **BreakIntoImages**(\mathbf{a}, n) breaks f into the images f_i . The procedure effectively has three sections. In the first section of the algorithm, we

Procedure AddContributions($\mathbf{a}, n, i, \text{add?}$)

Input:

- $n = \sum_{j=1}^s n_j$.
- \mathbf{a} , a length- n array containing f_1^*, \dots, f_{i-1}^* and $2r_i$, in that order.
- add? , a boolean value.

Result: The contribution of f_1^*, \dots, f_{i-1}^* towards $2f_i^*$ are added to $2r_i$, or subtracted if add? is false. As a result we will have $2f_i^*$ in place of $2r_i$.

```

1 if add? then c ← 1 else c ← -1
2 outOffset ←  $\sum_{j=1}^{i-1} n_j$ 
3 for j ← 1 to i-1 do
    // Add contribution of  $f_j^*$  to  $f_i^*$ 
4     inOffset ←  $\sum_{k=1}^{j-1} n_k$ 
5     for e ← 0 to  $n_j - 1$  do
6         if  $e[n(l)] = 0$  for  $j < l < i$  then
7              $\mathbf{a}(\text{outOffset} + (e \bmod n_i)) \mathrel{+}= c(-1)^{e[n(i)]} \mathbf{a}(\text{inOffset} + e)$ 

```

Procedure BreakIntoImages(\mathbf{a}, n): An in-place algorithm to compute a vector of images of f

Input: \mathbf{a} , a length- n array containing the coefficients of $f = \sum_{i=0}^{n-1} a_i z^i$, where $n = \sum_{i=1}^s n_i \cdot s$

Result: The images f_i , $1 \leq i \leq s$, are written in place of f .

```

// Write  $r_1, \dots, r_s$  in place of  $f$ 
1 m ← 0
2 for i ← 1 to s do
3     for j ← m to  $n - n_i - 1$  do  $\mathbf{a}(j) \leftarrow \mathbf{a}(j) - \mathbf{a}(j + n_i)$ 
4     m ← m +  $n_i$ 

// Write the weighted image  $f_i^*$  in place of  $r_i$ , for  $1 \leq i \leq s$ 
5 m ← 0
6 for i ← 2 to s do
7     for j ← m to  $m + n_i - 1$  do  $\mathbf{a}(j) \leftarrow 2\mathbf{a}(j)$ 
8     AddContributions( $\mathbf{a}, n, i, \text{true}$ )
9     for j ← m to  $m + n_i - 1$  do  $\mathbf{a}(j) \leftarrow \frac{1}{2}\mathbf{a}(j)$ 
10    m ← m +  $n_i$ 

// Reweigh  $f_i^*$  to get  $f_i$ 
11 m ←  $n_1$ 
12 for i ← 2 to s do
13     for j ← m to  $n - 1$  do  $\mathbf{a}(j) \leftarrow 2\mathbf{a}(j)$ 
14     m ← m +  $n_i$ 

```

break f into the remainders r_i . Producing r_i entails $n_i^* < n_i$ additions, and so producing all the r_i entails less than $\sum_{i=1}^s n_i = n$ ring additions.

In the second section we write the weighted images f_i^* in place of the r_i . Adding all the contributions, per lemma 8, requires $2n$ additions. We reweigh the last $n - n_1$ coefficients of f by 2, then by $\frac{1}{2}$. This constitutes less than n such multiplications.

In the third section we reweigh the weighted images f_i^* to get the images f_i . This entails less than n multiplications by 2. This gives us the following complexity:

Lemma 9. *Procedure BreakIntoImages(\mathbf{a}, n) entails no more than $3n$ additions and $2n$ multiplications by $2^{\pm 1}$.*

2.4 Putting the algorithm together

Procedure TFT(\mathbf{a}, n, ω): An in-place algorithm to compute a vector of evaluations of f

Input:

- \mathbf{a} , a length- n array containing the coefficients of $f = \sum_{i=0}^{n-1} a_i z^i$, where $n = \sum_{i=1}^s n_i$.
- ω , a root of Φ_1 .

Result: $\text{TFT}_{\omega}^{I(n)}(f)$ is written in place of f .

```

1 BreakIntoImages( $\mathbf{a}, n$ )
2  $(\gamma, m) \leftarrow (\omega, 0)$ 
3 for  $i \leftarrow 1$  to  $s$  do
4   HalfDFT( $\mathbf{a} + m, n_i, \gamma$ )
5    $m \leftarrow m + n_i$ 
6   if  $i < s$  then // Set  $\gamma$  to root of  $\Phi_{i+1}$ 
7     for  $j \leftarrow \log(n_{i+1})$  to  $\log(n_i) - 1$  do  $\gamma \leftarrow \gamma^2$ 

```

$\text{TFT}(\mathbf{a}, n, \omega)$ gives a detailed description of the entire TFT algorithm. The elements in arrays \mathbf{a} , and the elements γ and ω comprise the ring elements in the algorithm. To introduce array notation, given an array \mathbf{a} and integer m we will let $\mathbf{a} + m$ denote the array \mathbf{b} where $\mathbf{b}(i) = \mathbf{a}(i + m)$ for $i \geq 0$.

We note that, in order to make the algorithm truly in-place, we cannot store all the values n_i for $1 \leq i \leq s$. As the values n_i are always accessed sequentially, we could store the values n, n_1 , and n_s , and have functions which, given n and n_i , produces n_{i-1} and n_{i+1} respectively. This could be done by way of bit shifts and bit masks.

Theorem 10. $\text{TFT}(\mathbf{a}, n, \omega)$ entails fewer than

- $\frac{1}{2}n \log n + \mathcal{O}(n)$ ring multiplications,

- $n \log n + \mathcal{O}(n)$ ring additions, and
- $2n$ multiplications by $2^{\pm 1}$.

Proof. The complexity of *BreakIntoImages* was given by lemma 9. The brunt of the work is due to the half-DFTs. Counting the number of multiplications due to the half-DFTs by way of lemma 5 will give us

$$\sum_{i=1}^s \left[\frac{1}{2} n_i \log n_i + \mathcal{O}(n_i) \right] < \frac{1}{2} n \log n + \mathcal{O}(n), \quad (43)$$

as $\log n_i < \log n$ and $\sum_{i=1}^s n_i = n$. A similar analysis gives us $n \log n + \mathcal{O}(n)$ ring additions. \square

2.5 Obtaining a polynomial from its weighted TFT

We need a means of reversing the algorithm: constructing the coefficients of f from its evaluation points. Every step of the forward transform algorithm is reversible. We can reobtain f as follows:

1. Reobtain f_i from its respective half-DFT, for $1 \leq i \leq s$ by **InverseHalfDFT**. Reweigh each f_i to obtain f_i^* .
2. For $i = s$ down to $i = 1$, subtract all the contributions from f_i to obtain r_i .
3. Recall that q_{s-1} is exactly r_s . For $i = s - 1$ down to 1, write q_{i-1} in place of q_i and r_i , per the method described in section 2.2. The resulting polynomial is $q_0 = f$.

PutImagesBackTogether(\mathbf{a}, n) will, given an array \mathbf{a} containing the weighted images f_1, \dots, f_s , write f to the array \mathbf{a} in place of its weighted images. It has complexity comparable to *BreakIntoImages*. The function *InverseTFT* inverts a TFT, with the same stated complexity as **TFT**(\mathbf{a}, n, ω).

3 In-place polynomial multiplication

We present an algorithm for in-place multiplication of polynomials. In this algorithm, the input polynomials are preserved, and the only working space is an array \mathbf{c} allocated for the output product, and space for an additional $\mathcal{O}(1)$ ring elements and $\mathcal{O}(1)$ integers bounded by n . The algorithm mimics the structure of the Harvey-Roche algorithm for in-place polynomial multiplication [4], but instead uses the TFT algorithm presented here.

Let $f, g \in \mathcal{R}[z]$, and let $n = \deg(f) + \deg(g) + 1$, and let s, n_i, ω_i , and Φ_i , $1 \leq i \leq s$ be as defined in section 2. The algorithm will write $\text{TFT}_{\omega_1}^{I(n)}(fg)$ to the array \mathbf{c} , and then obtain $h = fg$ by performing an inverse-TFT on the output array.

Procedure PutImagesBackTogether(\mathbf{a}, n): An in-place algorithm to compute f given its images

Input: \mathbf{a} , a length- n array containing the images f_1, \dots, f_s .

Result: The polynomial f is written in place of its weighted images.

```

// Reweigh the images  $f_i$  to get  $f_i^*$ , for  $1 \leq i \leq s$ 
1  $m \leftarrow n_1$ 
2 for  $i \leftarrow 2$  to  $s$  do
3   for  $j \leftarrow m$  to  $n - 1$  do  $\mathbf{a}(j) \leftarrow \frac{1}{2}\mathbf{a}(j)$ 
4    $m \leftarrow m + n_i$ 

// Write  $r_i$  in place of  $f_i^*$ , for  $i = s, s-1, \dots, 1$ 
5  $(i, m) \leftarrow (s, n)$ 
6 while  $i > 0$  do
7    $m \leftarrow m - n_i$ 
8   for  $j \leftarrow m$  to  $m + n_i - 1$  do  $\mathbf{a}(j) \leftarrow 2\mathbf{a}(j)$ 
9   AddContributions( $\mathbf{a}, n, i, \text{false}$ )
10  for  $j \leftarrow m$  to  $m + n_i - 1$  do  $\mathbf{a}(j) \leftarrow \frac{1}{2}\mathbf{a}(j)$ 

// Write  $f$  in place of the remainders  $r_i$ 
11  $(i, m) \leftarrow (s, n)$ 
12 while  $i > 0$  do
13    $m \leftarrow m - n_i$ 
14   for  $j \leftarrow m$  to  $n - n_i - 1$  do  $\mathbf{a}(j) \leftarrow \mathbf{a}(j) + \mathbf{a}(j + n_i)$ 

```

Procedure InverseTFT(\mathbf{a}, n, ω): An in-place algorithm to obtain f from $\text{TFT}_{\omega}^{I(n)}(f)$

Input:

- \mathbf{a} , a length- n array containing $\text{TFT}_{\omega}^{I(n)}(f)$, where $n = \sum_{i=1}^s n_i$.
- ω , a root of Φ_1 .

Result: f is written in place of $\text{TFT}_{\omega}^{I(n)}(f)$.

```

1  $(\gamma, m) \leftarrow (\omega, 0)$ 
2 for  $i \leftarrow 1$  to  $s$  do
3   InverseHalfDFT( $\mathbf{a} + m, n_i, \gamma$ )
4    $m \leftarrow m + n_i$ 
5   if  $i < s$  then // Set  $\gamma$  to root of  $\Phi_{i+1}$ 
6     for  $j \leftarrow \log(n_{i+1})$  to  $\log(n_i) - 1$  do  $\gamma \leftarrow \gamma^2$ 
7 PutImagesBackTogether( $\mathbf{a}, n$ )

```

Let $h_i(z) = h(z) \bmod \Phi_i$, $1 \leq i \leq s$, be the images of $h = fg$. We will iteratively write the half-DFTs $\text{HDFT}_{\omega_i}^{n_i}(h)$ into the output array, giving us $\text{TFT}_{\omega_1}^{I(n)}(h)$. So as not to require more space, these half-DFTs will be computed in chunks of non-increasing, power-of-two sizes.

We suppose that $\text{HDFT}_{\omega_j}^{n_j}(h_j)$ has already been written in order to the front of output array for $1 \leq j < i$. Let \mathbf{c} be the array comprising the remaining $\sum_{j=i}^s n_i$ free space in our output array.

To write $\text{HDFT}_{\omega_i}^n(fg)$ to \mathbf{c} , we will let l be the least power of two such that we have $2l$ free array elements remaining. We handle the case where we have only one free array element separately. We will write length- l sections of $\text{HDFT}_{\omega_i}^{n_i}(f)$ and $\text{HDFT}_{\omega_i}^{n_i}(g)$ to unused space in \mathbf{c} , and then compute their pointwise product in place of the section of $\text{HDFT}_{\omega_i}^{n_i}(f)$, after which the length- l section of $\text{HDFT}_{\omega_i}^n(g)$ can be discarded and its array space reused.

If $i < s$, the sections will be of length $l = n_i/2, n_i/4, \dots, n_{i+1}$, and then we will compute a last additional length- n_{i+1} section. If $i = s$, we will compute sections of length $n_s/2, n_s/4, \dots, 1$, plus an additional length-1 section that is computed using auxiliary space for an additional ring element. Computing each section of the half-DFT will entail one pass through our inputs f and g . Thus in total over the entire multiplication algorithm we will pass through the input arrays $\log(n) + s$ times.

Fix $0 \leq k \leq \log(n_i)$ and define, for $1 \leq j \leq k$,

$$\Phi_{i,j} = z^{n_i/2^j} - \omega_i^{n_i/2^j} \quad \text{and} \quad \Phi_i^* = z^{n_i/2^k} - \omega_i^{-n_i/2^k}. \quad (44)$$

Then $\Phi_i = z^{n_i} + 1$ factors as

$$\left(\prod_{j=1}^k \Phi_{i,j} \right) \Phi_i^*. \quad (45)$$

For $i < s$ we will choose $k = \log(n_i - n_{i+1})$ and if $i = s$ we will let $\log(n_s)$. Moreover, the roots of $\Phi_{i,j}$ are of the form $z = \rho_j \gamma_j$, where $\rho_j = \omega_i^{2^j-1}$, and γ_j is an $(n_i/2^j)^{\text{th}}$ root of unity. Thus, to compute the section of $\text{HDFT}_{\omega_i}^{n_i}(f)$ (and similarly for g), comprising the evaluation of f at the roots of $\Phi_{i,j}$, we will first compute $f' = f(\rho_j z) \bmod (z^{n_i/2^j} - 1)$, and then compute $\text{DFT}_{\gamma_j}^{n_i/2^j}(f')$. A root of Φ_i^* will be of the form $z = \rho_{k+1} \gamma_k$, and so the evaluation of f and g at the roots of Φ_i^* can be computed similarly.

We leave it as an exercise to the reader to check that this will produce the evaluation of h at the roots of $z^{n_i} + 1$ in the same order as procedure $\text{HalfDFT}(\mathbf{c}, n_i, \omega_i)$.

After we have computed $\text{TFT}_{\omega_1}^{I(n)}(h)$, we then can perform the inverse TFT to give us $h(z)$. Algorithm 1 describes the multiplication algorithm.

Procedure ComputeHalfDFTofProduct($f, g, i, \omega_i, \mathbf{c}$)

Input: ;

- $f, g \in \mathcal{R}[z]$, two polynomials such that $\deg(fg) = n - 1$, where n can be written as a sum of decreasing powers of two $n = \sum_{i=1}^s n_i$.
- i , an integer such that $1 \leq i \leq s$.
- ω_i , a fixed root of $\Phi_i = z^{n_i} + 1$.
- \mathbf{c} , a length- n array, where $\sum_{j=i}^s n_j$, comprising elements of \mathcal{R} .

Result: $\text{HDFT}_{\omega_i}^n(fg)$ is written to the array \mathbf{c}

```

1 (freeSpace, offset)  $\leftarrow$   $(\sum_{j=i}^s n_j, 0)$ 

    // Evaluate at roots of  $\Phi_{i,j}$  and, if  $i < s$ ,  $\Phi_i^*$ 
2  $j \leftarrow 1$ 
3 while freeSpace > 1 do
4      $l \leftarrow 2^{\lfloor \log(\text{freeSpace}) \rfloor - 1}$ 
5      $(\rho, \gamma, j) \leftarrow (\omega_i^{2^j - 1}, \omega_i^{2n_i/l}, j + 1)$ 
6     Write  $f(\rho z) \bmod z^l - 1$  to  $\mathbf{c}(m), \dots, \mathbf{c}(\text{offset} + l - 1)$ 
7     Write  $g(\rho z) \bmod z^l - 1$  to  $\mathbf{c}(m + l), \dots, \mathbf{c}(\text{offset} + 2l - 1)$ 
8     FFT( $\mathbf{c} + \text{offset}, l, \gamma$ )
9     FFT( $\mathbf{c} + \text{offset} + l, l, \gamma$ )
10    for  $u \leftarrow 0$  to  $l - 1$  do
11         $\mathbf{c}(\text{offset} + u) \leftarrow \mathbf{c}(\text{offset} + u)\mathbf{c}(\text{offset} + l + u)$ 
12    freeSpace  $\leftarrow$  freeSpace -  $l$ 
13    offset  $\leftarrow$  offset +  $l$ 

    // Evaluate at roots of  $\Phi_s^*$  if  $i = s$ 
14 if freeSpace == 1 then
15      $\rho \leftarrow \omega_i^{2^{k+1} - 1}$ 
16      $\mathbf{c}(n - 1) \leftarrow (f(\rho z) \bmod z - 1)(g(\rho z) \bmod (z - 1))$ 

```

Algorithm 1: In-place polynomial multiplication

Input: ;

- $f, g \in \mathcal{R}[z]$, two polynomials such that $\deg(fg) = n - 1$, where n can be written as a sum of decreasing powers of two $n = \sum_{i=1}^s n_i$.
- \mathbf{c} , an array of size n .
- ω , a root of $z^{n_1} + 1$.

Result: The coefficients of $h = fg$ are written to \mathbf{c}

```

1  $m \leftarrow 0$ 
2 for  $i \leftarrow 1$  to  $s$  do
3     ComputeHalfDFTofProduct( $f, g, i, \omega^{n_1/n_i}, \mathbf{c} + m$ )
4      $m \leftarrow m + n_i$ .
5 InverseTFT( $\mathbf{c}, n, \omega$ )

```

4 Implementation

As a proof-of-concept, we implemented the WTFT algorithm, its inverse algorithm, and the algorithm for in-place multiplication via the WTFT. This implementation is a module for the **Maple** computer algebra system. Our implementation was done for polynomials over $\mathbb{Z}/p\mathbb{Z}$, where $p = 2013265921 = 2^{27} \cdot 15 + 1$. Our implementation cheats in that we store the values n_i , $1 \leq i \leq s$ in a list. There may potentially be $\lceil \log n \rceil$ such values. This implementation, written in a Maple worksheet, is made available at `cs.uwaterloo.ca/~a4arnold/code/tft.mw`.

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